Lecture 5: Ramsey-Cass-Koopmans Model

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Outline

- A brief review of the Diamond model
- Basic setup of the Ramsey-Cass-Koopmans (RCK) model
  - Individuals
  - Firms
  - Markets
  - Timing of events
- Competitive equilibrium (Solution) of the RCK model
  - Firm's profit maximisation
  - Individual's utility maximisation
  - Market clearing condition
  - Transition equations and stationary equilibrium

A Brief Review of the Diamond Model (1 of 6)

- Basic setup: OLG model, initial old, future generations
- Competitive equilibrium
  - Firm's profit maximisation problem

$$r_t = f'(k_t)$$
$$w_t = f(k_t) - f'(k_t)k_t$$

  - Individual's utility maximisation problem: Find a young individual's saving, s_t. The solution is then characterised by the Euler equation and the budget constraints:

$$\frac{u'(c_{1t})}{\beta u'(c_{2t+1})} = 1 + r_{t+1}$$

$$c_{1t} = w_t A_t - s_t \quad \text{and} \quad c_{2t+1} = (1 + r_{t+1})s_t$$

A Brief Review of the Diamond Model (2 of 6)

- For the CRRA utility

$$s_t = s(r_{t+1}) w_t A_t$$

- Market clearing

$$L_t^D = L_t$$
$$K_{t+1} = L_t s_t$$

- Equilibrium condition-transition equation for k_t combining the equation that characterise firm's and individual's problems and the market clearing conditions:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s[f'(k_{t+1})][f(k_t) - k_t f'(k_t)]$$

- A steady state value of k_t is some k^* such that $k_{t+1} = k_t = k^*$ satisfying the transition equation.

A Brief Review of the Diamond Model (3 of 6)

Is the competitive equilibrium Pareto-efficient?

- We examined the efficiency of the competitive equilibrium by comparing k^* with k_{GR}.
  - To find k_{GR}, write down the resource constraint for the Diamond economy:

$$F(K_t, A_t L_t) = L_t c_{1t} + L_{t-1} c_{2t} + (K_{t+1} - K_t)$$

  - Let $A_t = A$ for all t, the stationary resource constraint in per worker term is:

$$f(k) - nk = \frac{c_1}{A} + \frac{c_2}{A(1+n)}$$

- The golden-rule k maximises steady state consumption. Therefore, k_{GR} solves

$$\max_k f(k) - nk$$

  i.e. k_{GR} is determined by

$$f'(k_{GR}) = n$$

A Brief Review of the Diamond Model (4 of 6)

- k_{GR} can also be found by solving a social planner's problem, which chooses a stationary feasible allocation to maximise the welfare of future generations:

$$\max_{c_1, c_2, k} U(c_1, c_2) \quad s.t. \quad f(k) - nk = c_1 + \frac{c_2}{1+n}$$

- The golden rule allocation is Pareto-efficient, since it maximises future generations' utility among all feasible allocations. All other feasible allocations would give a lower welfare for future generations.

A Brief Review of the Diamond Model (5 of 6)

- The equilibrium allocation may not be the golden-rule allocation. Since k^* may not equal k_{GR}.
  - If $k^* = k_{GR}$, the competitive equilibrium coincides with the golden-rule allocation. Therefore, the equilibrium is efficient.
  - If $k^* < k_{GR}$, there is under accumulation of capital. The equilibrium allocation is Pareto-efficient.
  - If $k^* > k_{GR}$, there is over accumulation of capital. The equilibrium allocation is NOT Pareto-efficient, since a pay-as-you-go social security would make all generations better off.

A Brief Review of the Diamond Model (6 of 6)

- The inefficiency stems from the dynamic population structure of the model-dynamic inefficiency.
  - With finite horizon, the pay-as-you-go social security would hurt the last generation's welfare. i.e. it is not welfare improving over the equilibrium outcome
  - To summarise, if $k^* < k_{GR}$, the (stationary) competitive equilibrium (or the balanced growth path) is dynamically efficient; if $k^* > k_{GR}$, the (stationary) competitive equilibrium is dynamically inefficient.

Rationale of RCK Model

- Problem with the Solow model: ad-hoc assumption of constant saving rate.
- Will conclusions of Solow model be altered if saving is endogenously determined by utility maximisation?
  - Yes, but we will learn a lot about consumption/saving behavior by analysing it.
- Basic setup of Ramsey model was described by Ramsey in 1928.
- Dynamics were developed by Cass and Koopmans in a growth context in 1965.

Basic Setup of RCK Model (1 of 3)

- **Time** is discrete and the time horizon is infinite, $t = 0, 1, 2, \ldots$
- **Firms**:
  - Firms have access to a constant return to scale production function:

$$Y_t = F(K_t, A_t L_t^D) \tag{1}$$

    The intensive form $y_t = F(k_1, 1) \equiv f(k_t)$, where $y_t = \frac{Y_t}{A_t L_t^D}$, $k_t = \frac{K_t}{A_t L_t^D}$. f satisfies $f' > 0$, $f'' < 0$ (i.e. concave), $\lim_{k \to 0} f'(k) = +\infty$. There is no depreciation of capital.
  - Technology A grows at an exogenous rate g as in Solow-Swan model.
- **Individuals**:
  - The economy is populated with a fixed number L of identical individuals who live forever (i.e. no population growth).
  - We will show the number of individuals is inessential, thus can summarise individuals into representative individual.

Basic Setup of RCK Model (2 of 3)

- **Preference**: An individual's lifetime utility is given by:

$$U(c_0, c_1, ...) = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2}$$

  - U is additively separable.
  - The periodic utility function u satisfies $u' > 0$, $u'' < 0$ (i.e. concave), and $\lim_{c \to 0} u'(c) = +\infty$.
  - $0 < \beta < 1$ is the discount factor.
- **Endowment**:
  - Each individual is endowed with 1 unit of labour in each period, which is supplied inelastically to firms.
  - In addition, individuals are endowed with the initial stock of capital K_0 equally, which they rent to firms and may augment through saving.

Basic Setup of RCK Model (3 of 3)

- **Markets** are perfectly competitive. Denote the real interest rate (capital rental rate) and the wage rate per unit of effective labour in period t as r_t and w_t, respectively.
- **Initial conditions**: K_0, A_0, L are given.
- **Timing** of events in period t:
  - Firms hire labour and rent capital from individuals to produce output, sell the output in goods market, and pay individuals.
  - Individuals divide their wealth (labour income, capital income, and remaining savings) into consumption and savings (holding capital), then carry savings to period $t+1$.

Competitive Equilibrium (Solution) of RCK Model

- Conditions characterising **firm's profit maximisation**:

$$r_t = f'(k_t) \tag{3}$$

$$w_t = f(k_t) - f'(k_t)k_t \tag{4}$$

- **Individual's utility maximisation**:
  - Denote c_t and s_t as consumption and savings of an individual in period t (notice that c_t and s_t denote consumption and savings per worker).
  - Then an individual's budget constraint in period t is given by:

$$c_t + s_t = w_t A_t + (1 + r_t)s_{t-1}$$

For $t = 0$, $s_{-1} = \frac{K_0}{L}$, which are given.

Individual Optimisation Problem (1 of 4)

- An individual chooses a sequence of savings to maximise his/her lifetime utility, subject to budget constraints in each period, taking as given prices and technology levels.

$$\max_{\{s_t\}_{t=0}^{\infty}} U(c_0, c_1, ...) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad c_t + s_t = w_t A_t + (1 + r_t) s_{t-1}, \quad t = 0, 1, ... \tag{5}$$

$$s_{-1} = \frac{K_0}{L}$$

- Note that we only need to solve for $\{s_t\}_{t=0}^{\infty}$, then $\{c_t\}_{t=0}^{\infty}$ are determined from the budget constraints:

$$c_0 = w_0 A_0 + (1 + r_0) s_{-1} - s_0$$
$$c_1 = w_1 A_1 + (1 + r_1) s_0 - s_1$$
$$c_2 = w_2 A_2 + (1 + r_2) s_1 - s_2$$
$$...$$

Individual Optimisation Problem (2 of 4)

- This is a maximisation problem with infinite number of choice variables $\{s_t\}_{t=0}^{\infty}$ and infinite number of constraints. But it is not hard to work out the FOC with respect to a specific s_t.
- Differentiate the objective function w.r.t. s_t:

$$\frac{\partial U}{\partial s_t} = \sum_{t=0}^{\infty} \beta^t u'(c_t) \frac{\partial c_t}{\partial s_t}$$

- Notice that from the budget constraints, s_t only appears in c_t and c_{t+1}:

$$c_t = w_t A_t + (1 + r_t) s_{t-1} - s_t$$
$$c_{t+1} = w_{t+1} A_{t+1} + (1 + r_{t+1}) s_t - s_{t+1}$$

Individual Optimisation Problem (3 of 4)

- Therefore only $\frac{\partial c_t}{\partial s_t}$ and $\frac{\partial c_{t+1}}{\partial s_t}$ are nonzero. Then:

$$\begin{aligned}\frac{\partial U}{\partial s_t} &= \beta^t u'(c_t)\frac{\partial c_t}{\partial s_t} + \beta^{t+1} u'(c_{t+1})\frac{\partial c_{t+1}}{\partial s_t} \\ &= \beta^t u'(c_t)(-1) + \beta^{t+1} u'(c_{t+1})(1 + r_{t+1}) = 0\end{aligned}$$

- From the above equation, we can obtain:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1} \tag{6}$$

- Eq.(6) is the consumption Euler equation, which has a similar form as the Euler equation in the Diamond model. The individual's problem is characterised by Eq.(5) and (6) (budget constraint and Euler equation).

Individual Optimisation Problem (4 of 4)

- Substituting c_t and c_{t+1} into the Euler equation (6) , we have:

$$\frac{u'(w_t A_t + (1+r_t)s_{t-1} - s_t)}{\beta u'(w_{t+1}A_{t+1} + (1+r_{t+1})s_t - s_{t+1})} = 1 + r_{t+1}$$

- With s_{-1} is given, the optimal sequence of savings $\{s_t\}_{t=0}^{\infty}$ must satisfy the second-order difference equation above. Recall that in the Diamond model, s_t is determined by:

$$\frac{u'(w_t A_t - s_t)}{\beta u'((1+r_{t+1})s_t)} = 1 + r_{t+1}$$

which is one equation in one unknown: s_t

Market Clearing Condition

- Labour market:

$$L_t^D = L \tag{7}$$

- Capital Market:

$$K_{t+1} = Ls_t \tag{8}$$

Transition Equation (1 of 2)

- Need to combines conditions characterising firm's problem, individual's problem and market clearing conditions (Eq.(3) to (8)) to derive transition equations.
- Starting from market clearing condition for capital market: Eq.(8) implies that

$$s_t = \frac{K_{t+1}}{L} = \frac{A_{t+1}Lk_{t+1}}{L} = A_{t+1}k_{t+1} = (1+g)A_t k_{t+1} \tag{9}$$

- Substituting Eq.(3), (4) and (9) into the budget constraint (5), get:

$$\begin{aligned} c_t + (1+g)A_t k_{t+1} &= w_t A_t + (1+r_t)A_t k_t \\ c_t &= A_t[w_t + (1+r_t)k_t - (1+g)k_{t+1}] \\ \frac{c_t}{A_t} &= f(k_t) - f'(k_t)k_t + (1+f'(k_t))k_t - (1+g)k_{t+1} \\ \tilde{c}_t \equiv \frac{c_t}{A_t} &= f(k_t) + k_t - (1+g)k_{t+1} \end{aligned} \tag{10}$$

Notice that $\tilde{c}_t$ is the consumption per unit of effective labour. Then $c_t = A_t\tilde{c}_t$.

Transition Equation (2 of 2)

- Substituting this expression and Eq.(3) into the Euler equation (6):

$$\frac{u'(A_t\tilde{c}_t)}{u'(A_{t+1}\tilde{c}_{t+1})} = \beta[1 + f'(k_{t+1})] \tag{11}$$

- Eq.(10) and (11) are the transition equations of the system, which describe how capital per unit of effective labour (k_t) and consumption per unit of effective labour ($\tilde{c}_t$) evolve over time.

Example

- Assume utility function is CRRA, then Eq.(11) becomes:

$$\frac{(A_t\tilde{c}_t)^{-\theta}}{(A_{t+1}\tilde{c}_{t+1})^{-\theta}} = (\frac{A_{t+1}\tilde{c}_{t+1}}{A_t\tilde{c}_t})^{\theta} = (1+g)^{\theta}(\frac{\tilde{c}_{t+1}}{\tilde{c}_t})^{\theta} = \beta[1+f'(k_{t+1})]$$

$$(\frac{\tilde{c}_{t+1}}{\tilde{c}_t})^{\theta} = \frac{\beta[1+f'(k_{t+1})]}{(1+g)^{\theta}} \tag{12}$$

- For CRRA utility function, the transition equations are Eq.(10) and (12).

Steady State (1 of 2)

- A steady state of k_t and $\tilde{c}_t$, denoted as k^* and $\tilde{c}^*$, satisfy:

$$\tilde{c}^* = f(k^*) + k^* - (1+g)k^* = f(k^*) - gk^* \tag{13}$$

$$(1+g)^\theta = \beta[1 + f'(k^*)] \tag{14}$$

- Notice that Eq.(13) and (14) determine a unique steady state value, k^* and $\tilde{c}^*$. Recall that in the Diamond model with CRRA utility, there may be multiple values of k^* (i.e. multiple steady states).
- The convergence to the steady state is saddle path stable (to be discussed on next lecture).

Steady State (2 of 2)

- Again, once k_t and $\tilde{c}_t$ converge to their steady states, the economy reaches its stationary equilibrium, or balanced growth path.
  - Prices r_t and w_t are constants. The saving rate ($\frac{Ls_t}{Y_t} = \frac{s_t}{\frac{Y_t}{L}}$) is a constant.
  - Real wage rate per worker, capital per worker and output per worker grow at rate g.
  - Individual's consumption and saving in a period ($c_t = A_t\tilde{c}^*$, $s_t = \frac{K_{t+1}}{L} = (1+g)A_t k^*$) grow at rate g.
- Notice that L is inessential to our analysis, thus can be normalised to 1, i.e. can summaries individuals into a single representative individual (or consumer,or household) in the RCK model. That is why such models are called representative agent models.